# Generalized Reaction and Unrestricted Variational Formulation of Cavity Resonators—Part I: Basic Theory

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***Abstract*—Based on the reciprocity theorem, the reaction concept in electromagnetic theory is generalized to the cases where both surface electric and magnetic currents overlap across boundaries, i.e., neither the E-field, nor H-field meets the continuity conditions. An improved systematic method is then developed to obtain unrestricted variational expressions in a cavity resonator for which the tangential components of the trial fields can be discontinuous across its interior boundaries.**

***Index Terms*—Bilinear form, cavity resonators, dielectric resonators, eigenvalue problem, mode-matching method, reaction, variational formulation.**

## I. Introduction

CAVITY resonators have been of wide-ranging use in microwave engineering and are fundamental building blocks of filters, multiplexers, and oscillators. Miniaturization and characteristic stabilization of these resonators can be achieved by loading the cavity with high-dielectric constant ceramics and making the so-called dielectric resonators (DRs) [1]–[3]. Obtaining cavity modes and resonant frequencies, which are essential in designing DR structures, are usually very complicated and time consuming. Various methods have been developed for these purposes [4]. Among them, the mode-matching method is the most widely used one [4]–[6]. For many applications, the resonant frequency should be calculated with less than 1% error. This, however, cannot be achieved by choosing only a few number of modes. On the other hand, including a large number of modes in the analysis makes the process very slow.

The strong motivation behind this theoretical work is to explore the possibility of increasing the accuracy of the calculation of resonant frequency by using variational expressions in the mode-matching method.

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Variational methods are well established in physical and engineering problems with solid foundations in physics and mathematics. The importance of eigenvalue theory in pure and applied mathematics, and in physics and their approximate calculations and the fact that variational methods are well adapted to successive approximation methods make this approach the very central fields of analysis. The applicability of this method depends on the availability of variational formulas. Variational schemes used in the analysis of DRs are addressed in [4] and [7]. Further, the reader may find a good review on variational methods in electromagnetic theory in [8, Ch. 5].

Based on the concept of reaction, which is defined by Rumsey [9], Harrington, in an excellent chapter, developed a systematic method for obtaining variational expressions for resonant frequencies of cavity resonators [10, pp. 340–345]. However, by the nature of development, one cannot handle the case where both trial electric and magnetic fields do not meet the continuity conditions across some boundaries inside the cavity. In this sense, even the most general form of the variational expressions obtained by using Harrington's approach are *restricted*. On the other hand, one should note that the fields obtained by the mode-matching method do not satisfy the restrictions set forth by Harrington's approach. More precisely, neither tangential components of the electric field, nor magnetic field obtained by using a finite number of modes satisfy the continuity conditions across the boundaries inside the cavity where the fields are enforced to match. Therefore, one needs stationary expressions that relax the boundary conditions of at least electric or magnetic field across the boundaries inside the cavity. In a classic paper, Berk tried to expand the class of trial fields to include discontinuous tangential electric and magnetic fields in a lossless cavity. However, the formula given by him is not correct [11, p. 105].

To make the class of trial fields unrestricted in the Harrington's approach, one should extend the reaction concept to the cases where both electric and magnetic surface currents coincide. In Section II, by using the reciprocity theorem, we will generalize the reaction concept in this sense. As will be shown, because of the discontinuity of tangential components of both electric and magnetic fields, the generalized reaction is not unique. However, it reduces to those conventional ones if at least the electric or magnetic field satisfies the boundary conditions at any interface inside the cavity. In Section III, after clarifying Harrington's approach, we will set up the machinery for developing unrestricted variational expressions inside cavity resonators. This improved systematic method is based

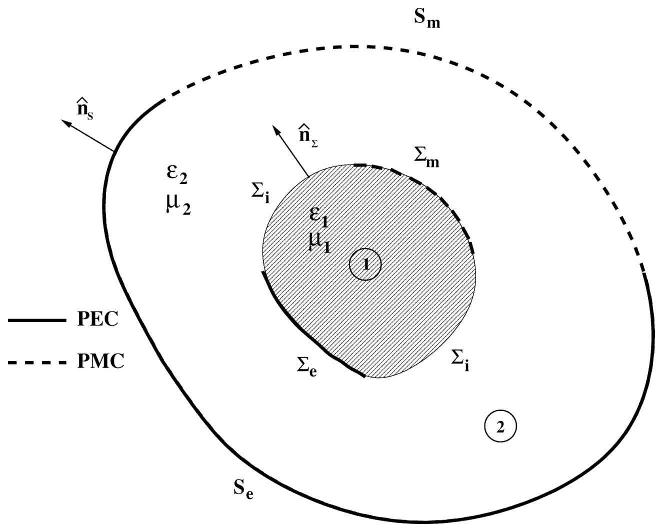


Fig. 1. Arbitrary cavity resonator.

on the degenerate or singular character (to be defined later) of the reaction in cavity resonators. This method is slightly different than Harrington's approach and is more meaningful. Since this systematic method is based on the generalized reaction, the variational expressions obtained on this basis are not unique. However, under special cases, i.e., restricted fields, they reduce to those given by Harrington. Section IV is devoted to numerical results and discussion. Finally, conclusions are summarized in Section V.

We have demonstrated that when a field obtained by the mode-matching method is used as a trial one in a particular form of the variational expressions, the resonant frequency does not change [12]. This behavior states that this solution is a stationary point of a variational expression and should be expected since the mode-matching method is equivalent to the Galerkin approach. We are then led to the generalize mode-matching method to nonorthogonal and free-boundary cases where a set of basis functions used for the field expansion inside a cavity resonator do not have to be orthogonal or satisfy any specific boundary condition. Details of this exploration is given in Part II of this paper [12].

## II. Generalized Reaction

Consider a cavity resonator bounded partly by a perfect electric conductor and partly by a perfect magnetic conductor, as illustrated in Fig. 1. For simplicity, we assume that the interior region of the cavity contains two homogeneous materials as illustrated. Moreover, assume that a part of the interface between the two media is partly covered by a perfect electric conductor and partly by a perfect magnetic conductor. Let $(\mathbf{E}^a, \mathbf{H}^a)$ and $(\mathbf{E}^b, \mathbf{H}^b)$ be two arbitrary sets of vector fields defined inside the cavity, which do not necessarily satisfy the boundary conditions within the cavity or on its walls. One may associate the electromagnetic fields $\mathbf{a}$ and $\mathbf{b}$, operating at a frequency $\omega$, with each set of vector fields, respectively, by finding some source that would generate the assumed field distributions. Each associated electromagnetic field is supported in part by volume sources, which can be found by vector operators acting on the assumed field distributions as

$$\mathbf{M}_{1,2}^{a,b} = -\nabla \times \mathbf{E}_{1,2}^{a,b} - j\omega\mu_{1,2}\mathbf{H}_{1,2}^{a,b} \quad (1)$$

$$\mathbf{J}_{1,2}^{a,b} = \nabla \times \mathbf{H}_{1,2}^{a,b} - j\omega\epsilon_{1,2}\mathbf{E}_{1,2}^{a,b}. \quad (2)$$

The rest of the sources are of surface type, which are necessary to support the boundary conditions within the cavity or on its walls. Before considering these sources, we apply the standard procedure used in the derivation of the reciprocity theorem [10, pp. 116–118] to get

$$\int_{V_1} \left(\mathbf{E}_1^a \cdot \mathbf{J}_1^b - \mathbf{H}_1^a \cdot \mathbf{M}_1^b - \mathbf{E}_1^b \cdot \mathbf{J}_1^a + \mathbf{H}_1^b \cdot \mathbf{M}_1^a\right) dv = \int_{\Sigma} \left(\mathbf{E}_1^b \times \mathbf{H}_1^a - \mathbf{E}_1^a \times \mathbf{H}_1^b\right) \cdot \hat{\mathbf{n}}_\Sigma \, ds \quad (3)$$

$$\int_{V_2} \left(\mathbf{E}_2^a \cdot \mathbf{J}_2^b - \mathbf{H}_2^a \cdot \mathbf{M}_2^b - \mathbf{E}_2^b \cdot \mathbf{J}_2^a + \mathbf{H}_2^b \cdot \mathbf{M}_2^a\right) dv = \int_{S} \left(\mathbf{E}_2^b \times \mathbf{H}_2^a - \mathbf{E}_2^a \times \mathbf{H}_2^b\right) \cdot \hat{\mathbf{n}}_S \, ds - \int_{\Sigma} \left(\mathbf{E}_2^b \times \mathbf{H}_2^a - \mathbf{E}_2^a \times \mathbf{H}_2^b\right) \cdot \hat{\mathbf{n}}_\Sigma \, ds \quad (4)$$

where $\Sigma = \Sigma_e + \Sigma_m + \Sigma_i$, $S = S_e + S_m \cdot \Sigma_e$, and $\Sigma_m$ are parts of the interior boundary made of perfect electric and perfect magnetic conductors, respectively. $\Sigma_i$ is just the interface between the two dielectric materials and $S_e$ and $S_m$, respectively, are parts of the cavity walls made of perfect electric and perfect magnetic conductors.

Now if $\hat{\mathbf{n}}_\Sigma \times \mathbf{E}_{1,2}^{a,b}(\mathbf{r}) \neq 0$ for $\mathbf{r} \in \Sigma_e$, one should place the magnetic surface currents $\mathbf{M}_{s_{\Sigma_{e_1}}}^{a,b} = \mathbf{E}_1^{a,b} \times (-\hat{\mathbf{n}}_\Sigma)$ on $\Sigma_e$ immediately inside region 1 and $\mathbf{M}_{s_{\Sigma_{e_2}}}^{a,b} = \mathbf{E}_2^{a,b} \times \hat{\mathbf{n}}_\Sigma$ on $\Sigma_e$ immediately inside region 2. The outer surface $S_e$ can be treated in the same way, whereas $\Sigma_m$ and $S_m$ must be treated in a dual fashion. Of particular importance is the surface $\Sigma_i$. To support the discontinuities of the tangential components of $\mathbf{E}^{a,b}$ and $\mathbf{H}^{a,b}$, both electric and magnetic surface currents must be added to this surface, i.e., $\mathbf{J}_{s_{\Sigma_i}}^{a,b} = \hat{\mathbf{n}}_\Sigma \times (\mathbf{H}_2^{a,b} - \mathbf{H}_1^{a,b})$ and $\mathbf{M}_{s_{\Sigma_i}}^{a,b} = (\mathbf{E}_2^{a,b} - \mathbf{E}_1^{a,b}) \times \hat{\mathbf{n}}_\Sigma$, respectively.

The final form of the generalized reaction can be obtained by adding both sides of (3) and (4) and transforming the integrand of each surface integral to the dot products of the fields and surface currents. This is not new as far as the surfaces $\Sigma_e, \Sigma_m, S_e$, and $S_m$ are concerned. For example, $\int_{\Sigma_e} \mathbf{E}_1^b \times \mathbf{H}_1^a \cdot \hat{\mathbf{n}}_\Sigma \, ds = \int_{\Sigma_e} \mathbf{H}_1^a \cdot \mathbf{M}_{s_{\Sigma_{e_1}}}^b \, ds$ and $\int_{\Sigma_m} \mathbf{E}_1^b \times \mathbf{H}_1^a \cdot \hat{\mathbf{n}}_\Sigma \, ds = \int_{\Sigma_m} \mathbf{E}_1^b \cdot \mathbf{J}_{s_{\Sigma_{m_1}}}^a \, ds$ and so on. However, the surface integral over $\Sigma_i$ cannot be treated in this simple way if the tangential components of $\mathbf{E}^a$ and $\mathbf{H}^a$ or $\mathbf{E}^b$ and $\mathbf{H}^b$ are discontinuous on $\Sigma_i$. More precisely, by adding the right-hand sides of (3) and (4), the surface integral over $\Sigma_i$ will be of the following form:

$$\int_{\Sigma_i} \triangleq \int_{\Sigma_i} \left(\mathbf{E}_1^b \times \mathbf{H}_1^a - \mathbf{E}_1^a \times \mathbf{H}_1^b - \mathbf{E}_2^b \times \mathbf{H}_2^a + \mathbf{E}_2^a \times \mathbf{H}_2^b\right) \cdot \hat{\mathbf{n}}_\Sigma \, ds. \quad (5)$$

On the other hand, the electric and magnetic surface currents that support the discontinuities of the $\mathbf{a}$ and $\mathbf{b}$-field systems on

$\Sigma_i$ are $\mathbf{J}^{a,b}_{s_{\Sigma_i}} = \hat{\mathbf{n}}_\Sigma \times (\mathbf{H}^{a,b}_2 - \mathbf{H}^{a,b}_1)$ and $\mathbf{M}^{a,b}_{s_{\Sigma_i}} = (\mathbf{E}^{a,b}_2 - \mathbf{E}^{a,b}_1) \times \hat{\mathbf{n}}_\Sigma$, respectively. Now it becomes clear that because of the discontinuity of the tangential components of the fields on $\Sigma_i$, the integrand in (5) cannot be easily converted to the dot products of the fields with $\mathbf{J}^{a,b}_{s_{\Sigma_i}}$ and $\mathbf{M}^{a,b}_{s_{\Sigma_i}}$. Despite this fact, by proper definition of $\mathbf{E}^{a,b}$ and $\mathbf{H}^{a,b}$ on $\Sigma_i$, we try to write (5) in the following form:

$$\int_{\Sigma_i} = \int_{\Sigma_i} \Big( -\mathbf{E}^a_{\Sigma_i} \cdot \mathbf{J}^b_{s_{\Sigma_i}} + \mathbf{H}^a_{\Sigma_i} \cdot \mathbf{M}^b_{s_{\Sigma_i}} + \mathbf{E}^b_{\Sigma_i} \cdot \mathbf{J}^a_{s_{\Sigma_i}} - \mathbf{H}^b_{\Sigma_i} \cdot \mathbf{M}^a_{s_{\Sigma_i}} \Big) ds. \quad (6)$$

Since we have assumed that the tangential components of $\mathbf{E}^{a,b}$ and $\mathbf{H}^{a,b}$ are discontinuous on $\Sigma_i, \mathbf{E}^{a,b}_{\Sigma_i}$ and $\mathbf{H}^{a,b}_{\Sigma_i}$ are as yet arbitrary on this surface and we may define them in such a way that (6) reduces to (5). At first, it seems natural to define $\mathbf{E}^{a,b}$ and $\mathbf{H}^{a,b}$ on $\Sigma_i$ as averages of the fields on both sides of $\Sigma_i$. However, as will be shown shortly, we can proceed even more general than that. Let us define $\mathbf{E}^{a,b}$ and $\mathbf{H}^{a,b}$ on $\Sigma_i$ as linear combinations of the fields on both sides of $\Sigma_i$ as follows:

$$\mathbf{E}^{a,b}_{\Sigma_i} = \alpha_1(\sigma)\mathbf{E}^{a,b}_1 + \alpha_2(\sigma)\mathbf{E}^{a,b}_2 \quad (7)$$
$$\mathbf{H}^{a,b}_{\Sigma_i} = \beta_1(\sigma)\mathbf{H}^{a,b}_1 + \beta_2(\sigma)\mathbf{H}^{a,b}_2 \quad (8)$$

where each coefficient is considered as a function of $\sigma$, the characterizing parameter of the surface $\Sigma_i$. The above coefficients are not independent and the relation between them can be obtained by substituting (7) and (8) into (6), and then equating the resulting integral with (5). However, before doing that, we can obtain the relations between $\alpha_1(\sigma)$ and $\alpha_2(\sigma)$ or $\beta_1(\sigma)$ and $\beta_2(\sigma)$ by a simple reasoning. More specifically, under special cases where the tangential components of the $\mathbf{E}$-field are continuous on $\Sigma_i$, we should have

$$\hat{\mathbf{n}}_\Sigma \times \mathbf{E}^{a,b}_{\Sigma_i} = \hat{\mathbf{n}}_\Sigma \times \mathbf{E}^{a,b}_1 = \hat{\mathbf{n}}_\Sigma \times \mathbf{E}^{a,b}_2.$$

A similar relation also holds for the magnetic field if the tangential components of the $\mathbf{H}$-field are continuous on $\Sigma_i$. Such relations require that

$$\alpha_2(\sigma) = 1 - \alpha_1(\sigma) \quad (9)$$
$$\beta_2(\sigma) = 1 - \beta_1(\sigma). \quad (10)$$

As will be explained later, (9) and (10) are also required for source conservation. To obtain (9) and (10) more rigorously, we substitute (7) and (8) into (6) and express the surface currents in terms of the fields. After simple algebraic manipulations, we get

$$\begin{aligned}\int_{\Sigma_i} = &\int_{\Sigma_i} [\alpha_1(\sigma) - \beta_2(\sigma)] \Big( \mathbf{E}^a_1 \times \mathbf{H}^b_2 - \mathbf{E}^b_1 \times \mathbf{H}^a_2 \Big) \cdot \hat{\mathbf{n}}_\Sigma \, ds \\ &+ \int_{\Sigma_i} [\beta_1(\sigma) - \alpha_2(\sigma)] \Big( \mathbf{E}^a_2 \times \mathbf{H}^b_1 - \mathbf{E}^b_2 \times \mathbf{H}^a_1 \Big) \cdot \hat{\mathbf{n}}_\Sigma \, ds \\ &+ \int_{\Sigma_i} [\alpha_1(\sigma) + \beta_1(\sigma)] \Big( \mathbf{E}^b_1 \times \mathbf{H}^a_1 - \mathbf{E}^a_1 \times \mathbf{H}^b_1 \Big) \cdot \hat{\mathbf{n}}_\Sigma \, ds \\ &+ \int_{\Sigma_i} [\alpha_2(\sigma) + \beta_2(\sigma)] \Big( -\mathbf{E}^b_2 \times \mathbf{H}^a_2 + \mathbf{E}^a_2 \times \mathbf{H}^b_2 \Big) \cdot \hat{\mathbf{n}}_\Sigma \, ds. \end{aligned} \quad (11)$$

On comparison of (11) with (5), $\alpha$'s and $\beta$'s should satisfy the following relations:

$$\alpha_1(\sigma) - \beta_2(\sigma) = \beta_1(\sigma) - \alpha_2(\sigma) = 0 \quad (12)$$
$$\alpha_1(\sigma) + \beta_1(\sigma) = \alpha_2(\sigma) + \beta_2(\sigma) = 1. \quad (13)$$

Using (12) and (13), one may come up with (9) and (10), which have been obtained earlier by an intuitive way. Now (7) and (8) can be written as

$$\mathbf{E}^{a,b}_{\Sigma_i} = \alpha(\sigma)\mathbf{E}^{a,b}_1 + \big[1 - \alpha(\sigma)\big]\mathbf{E}^{a,b}_2 \quad (14)$$
$$\mathbf{H}^{a,b}_{\Sigma_i} = \big[1 - \alpha(\sigma)\big]\mathbf{H}^{a,b}_1 + \alpha(\sigma)\mathbf{H}^{a,b}_2 \quad (15)$$

where $\alpha(\sigma) \triangleq \alpha_1(\sigma)$. According to (14) and (15), $\mathbf{E}^{a,b}(\mathbf{H}^{a,b})$ on $\Sigma_i$ are defined as a *quasi-convex* combination of $\mathbf{E}^{a,b}_1(\mathbf{H}^{a,b}_1)$ and $\mathbf{E}^{a,b}_2(\mathbf{H}^{a,b}_2)$. In contrast to convex combination, $\alpha(\sigma)$ is not required to satisfy $0 < \alpha(\sigma) < 1$. Therefore, we use the term "quasi." By defining $\mathbf{E}^{a,b}$ and $\mathbf{H}^{a,b}$ on $\Sigma_i$ by (14) and (15), (5) can be written in the same form as (6). Note that defining $\mathbf{E}^{a,b}$ and $\mathbf{H}^{a,b}$ on $\Sigma_i$ by averages of the fields on both sides of $\Sigma_i$ can be considered as a special case of (14) and (15) with $\alpha(\sigma) \equiv 1/2$. An interesting interpretation of (9) and (10) can be given by substituting (14) and (15) into (6). To this end, (6) reduces to

$$\begin{aligned}\int_{\Sigma_i} = &\int_{\Sigma_i} \Big( -\mathbf{E}^a_1 \cdot \alpha(\sigma)\mathbf{J}^b_{s_{\Sigma_i}} - \mathbf{E}^a_2 \cdot \big[1 - \alpha(\sigma)\big]\mathbf{J}^b_{s_{\Sigma_i}} \\ &\quad + \mathbf{H}^a_1 \cdot \big[1 - \alpha(\sigma)\big]\mathbf{M}^b_{s_{\Sigma_i}} + \mathbf{H}^a_2 \cdot \alpha(\sigma)\mathbf{M}^b_{s_{\Sigma_i}} \Big) ds \\ &+ \int_{\Sigma_i} \Big( \mathbf{E}^b_1 \cdot \alpha(\sigma)\mathbf{J}^a_{s_{\Sigma_i}} + \mathbf{E}^b_2 \cdot \big[1 - \alpha(\sigma)\big]\mathbf{J}^a_{s_{\Sigma_i}} \\ &\quad - \mathbf{H}^b_1 \cdot \big[1 - \alpha(\sigma)\big]\mathbf{M}^a_{s_{\Sigma_i}} - \mathbf{H}^b_2 \cdot \alpha(\sigma)\mathbf{M}^a_{s_{\Sigma_i}} \Big) ds. \end{aligned} \quad (16)$$

Equation (16) means that instead of defining the fields on the surface of the discontinuity, one may divide the surface currents between the two regions and take the dot products of the fields on each side of that surface with the corresponding sources. Thus, (9) and (10) are just based on the conservation of sources. Now if one uses (14) and (15) in (6), one can obtain the general form of the reaction by adding (3) and (4) and substituting (5) by (6). More precisely, by adding (3) and (4) and rearranging the terms, one may write

$$\langle \mathbf{a}, \mathbf{b} \rangle_\alpha = \langle \mathbf{b}, \mathbf{a} \rangle_\alpha \quad (17)$$

where

$$\begin{aligned}\langle \mathbf{a}, \mathbf{b} \rangle_\alpha \triangleq &\int_{V_1} \Big( \mathbf{E}^a_1 \cdot \mathbf{J}^b_1 - \mathbf{H}^a_1 \cdot \mathbf{M}^b_1 \Big) dv \\ &+ \int_{\Sigma_m} \mathbf{E}^a_1 \cdot \mathbf{J}^b_{s_{\Sigma_{m_1}}} ds - \int_{\Sigma_e} \mathbf{H}^a_1 \cdot \mathbf{M}^b_{s_{\Sigma_{e_1}}} ds \\ &+ \int_{\Sigma_i} \Big( \mathbf{E}^a_{\Sigma_i} \cdot \mathbf{J}^b_{s_{\Sigma_i}} - \mathbf{H}^a_{\Sigma_i} \cdot \mathbf{M}^b_{s_{\Sigma_i}} \Big) ds \\ &+ \int_{V_2} \Big( \mathbf{E}^a_2 \cdot \mathbf{J}^b_2 - \mathbf{H}^a_2 \cdot \mathbf{M}^b_2 \Big) dv \\ &+ \int_{\Sigma_m} \mathbf{E}^a_2 \cdot \mathbf{J}^b_{s_{\Sigma_{m_2}}} ds - \int_{\Sigma_e} \mathbf{H}^a_2 \cdot \mathbf{M}^b_{s_{\Sigma_{e_2}}} ds \\ &+ \int_{S_m} \mathbf{E}^a_2 \cdot \mathbf{J}^b_{s_{S_m}} ds - \int_{S_e} \mathbf{H}^a_2 \cdot \mathbf{M}^b_{s_{S_e}} ds. \end{aligned} \quad (18)$$

$\mathbf{E}^a_{\Sigma_i}$ and $\mathbf{H}^a_{\Sigma_i}$ are defined by (14) and (15), respectively, and $\langle \mathbf{b}, \mathbf{a} \rangle_\alpha$ can be obtained from $\langle \mathbf{a}, \mathbf{b} \rangle_\alpha$ by interchanging the superscripts $a$ and $b$ in (18). $\langle \mathbf{a}, \mathbf{b} \rangle_\alpha$ is called the reaction of field $\mathbf{a}$ on field $\mathbf{b}$. It should be emphasized that unlike the cases that have been treated in the literature, the generalized reaction is not unique and it depends on $\alpha(\sigma)$, which can be defined arbitrarily on any surface like $\Sigma_i$ inside the cavity. However, it will be independent of $\alpha(\sigma)$ if at least the electric or magnetic fields of both of the field systems meet the continuity conditions on $\Sigma_i$ and reduces to those conventional ones in the literature. It should also be noted that if region 2 in Fig. 1 is unbounded and both $\mathbf{a}$- and $\mathbf{b}$-field systems satisfy the radiation conditions, (17) without any surface integral over $S_e$ and $S_m$ is also valid.

We have defined $\mathbf{a}$-field and $\mathbf{b}$-field by assuming the $\mathbf{E}$- and $\mathbf{H}$-field distribution within the cavity and, since according to (1) and (2), the volume sources are expressed in terms of these fields, the reaction can also be expressed in terms of these assumed field distributions and (17) is, in fact, an identity between two sets of arbitrary vector fields and the assumed frequency. It is also possible to set up an electromagnetic field by assuming only electric-field distribution or only magnetic-field distribution within the cavity. In the former case, by setting $\mathbf{M}^{a,b}_{1,2} = 0$ in (1), the $\mathbf{H}$-field can be obtained in terms of the assumed $\mathbf{E}$-field distribution and, therefore, by using (2), $\mathbf{J}^{a,b}_{1,2}$ can also be expressed in terms of the $\mathbf{E}$-field. This means that the reaction can be expressed in terms of the assumed frequency and $\mathbf{E}$-field distribution. For setting up an electromagnetic field by assuming only magnetic-field distribution within the cavity, we set $\mathbf{J}^{a,b}_{1,2} = 0$ in (2) and proceed in a dual fashion. For a special case where the assumed field distributions are not supported by any volume sources, (17) reduces to an identity between the two defined vector fields within the cavity. These facts will be explored in more details later when we obtain various variational formulas.

For future reference, we consider the important case where the $\mathbf{b}$-field is one of the resonant modes of the cavity illustrated in Fig. 1. Let us denote this correct resonant field by $\mathbf{c}$ and reserve $\mathbf{a}$ for the approximate field, which is also defined at the same resonant frequency. Since the correct resonant field is source free, we have $\langle \mathbf{a}, \mathbf{c} \rangle_\alpha = 0$ and, according to (17), we end up with the following important relation:

$$\langle \mathbf{a}, \mathbf{c} \rangle_\alpha = \langle \mathbf{c}, \mathbf{a} \rangle_\alpha = 0. \tag{19}$$

It should be noted that since the $\mathbf{c}$-field satisfies all boundary conditions, the second term in (19) is independent of $\alpha$. Consequently, the first term is also independent of $\alpha$ (as it should be).

Equation (19) plays a key role in our discussion and is the fundamental relation that we will use later to derive unrestricted variational formulas.

In summary, in this section, we have derived the most general form of the reaction in which the tangential components of both electric and magnetic fields can be discontinuous across some interfaces. Recall that the generalized reaction defined in this section is not unique. As will be shown later, this fact can be exploited to obtain various unrestricted variational formulas inside cavity resonators.

## III. Basic Formulation

In Section II, it has been explained how to set up an electromagnetic field operating at an arbitrary frequency from a given arbitrary field distribution within a cavity. For a given $\alpha(\sigma)$, the class of such electromagnetic fields operating at a specific frequency is a linear space.

The so-called reaction originally defined by Rumsey [9] is a physical observable. This observable and its more general form defined in (18) has the mathematical structure of a *symmetric bilinear form* in an infinite-dimensional linear space of electromagnetic fields defined at a specific frequency within a cavity [13, p. 367]. In addition, reaction is *degenerate* (or *singular*) in the linear space of the electromagnetic fields defined at any *resonant* frequency within a cavity [13, p. 365]. By "degenerate" (or "singular"), we mean that there exits at least a nonzero element in the linear space of electromagnetic fields defined at some resonant frequency $\omega_r$ within a cavity resonator such that its reaction on all other electromagnetic fields defined at the same resonant frequency within that cavity is zero. Equation (19) states that the exact resonant field $\mathbf{c}$ is, in fact, such an element. In the literature, it is common to use a bracket for the inner product. Therefore, its use to express the reaction that is a bilinear form and mathematically is more general than the inner product is not proper and sometimes is confusing. However, since historically this notation was used for the reaction and it also appears in the literature, we use the same notation to represent it.

Despite the fact that reaction is not an inner product and one cannot define any norm or angle, it is useful to extend the concept of orthogonality as follows.

*Definition:* Two electromagnetic-field systems $\mathbf{a}$ and $\mathbf{b}$ defined at the same frequency are called orthogonal in the reaction sense if

$$\langle \mathbf{a}, \mathbf{b} \rangle_\alpha = \langle \mathbf{b}, \mathbf{a} \rangle_\alpha = 0.$$

Rumsey used the reaction concept to obtain approximate formulas for scattering coefficients, transmission coefficients, and aperture impedance by enforcing the approximate field $\mathbf{a}$, and the correct field $\mathbf{c}$ look the same to an arbitrary available test field $\mathbf{t}$ in the reaction sense, i.e., $\langle \mathbf{t}, \mathbf{a} \rangle = \langle \mathbf{t}, \mathbf{c} \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the conventional reaction [9]. As a special case, one may consider the approximate field as a test function and, therefore, one may enforce $\langle \mathbf{a}, \mathbf{a} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle$. Similarly, if one expresses the approximate field as a linear combination of $\mathbf{v}_i, (i = 1, 2, \ldots, N)$, one may also consider $\mathbf{v}_i$ as a test function and enforce $\langle \mathbf{v}_i, \mathbf{a} \rangle = \langle \mathbf{v}_i, \mathbf{c} \rangle, (i = 1, 2, \ldots, N)$. Since reaction is a bilinear form, the latter equality implies $\langle \mathbf{a}, \mathbf{a} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle$. Moreover, all the above constraints can be considered as enforcing the error field to be orthogonal to the available test functions in the reaction sense. More precisely, let $\mathbf{e}_a \triangleq \mathbf{a} - \mathbf{c}$ be the error field. $\langle \mathbf{t}, \mathbf{a} \rangle = \langle \mathbf{t}, \mathbf{c} \rangle, \langle \mathbf{a}, \mathbf{a} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle$ and $\langle \mathbf{v}_i, \mathbf{a} \rangle = \langle \mathbf{v}_i, \mathbf{c} \rangle, (i = 1, 2, \ldots, N)$ are then equivalent to $\langle \mathbf{t}, \mathbf{e}_a \rangle = 0, \langle \mathbf{a}, \mathbf{e}_a \rangle = 0$, and $\langle \mathbf{v}_i, \mathbf{e}_a \rangle = 0, (i = 1, 2, \ldots, N)$, respectively. It should be emphasized that orthogonality of the error field to the available test functions in the reaction sense does not imply least square approximation because no norm is defined by the reaction.

As mentioned above, the condition $\langle \mathbf{a}, \mathbf{a} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle$ means $\langle \mathbf{a}, \mathbf{e}_a \rangle = 0$, which indicates that $\mathbf{e}_a$ is perpendicular to $\mathbf{a}$ in the reaction sense. Since $\mathbf{c} = \mathbf{a} - \mathbf{e}_a$ and the reaction is a bilinear form, one may write $\langle \mathbf{c}, \mathbf{c} \rangle = \langle \mathbf{a}, \mathbf{a} \rangle + \langle \mathbf{e}_a, \mathbf{e}_a \rangle$, which can be interpreted as Pythagoras' theorem in the reaction. According to Pythagoras' theorem, one may write $\langle \mathbf{a}, \mathbf{a} \rangle = \langle \mathbf{c}, \mathbf{c} \rangle - \langle \mathbf{e}_a, \mathbf{e}_a \rangle$ or

$$\langle \mathbf{c} + \mathbf{e}_a, \mathbf{c} + \mathbf{e}_a \rangle = \langle \mathbf{c}, \mathbf{c} \rangle - \langle \mathbf{e}_a, \mathbf{e}_a \rangle. \tag{20}$$

Equation (20) proves the stationary character of $\langle \mathbf{c} + \mathbf{e}_a, \mathbf{c} + \mathbf{e}_a \rangle$ about $\mathbf{c}$. In fact, by defining $f(p) \triangleq \langle \mathbf{c} + p\mathbf{e}, \mathbf{c} + p\mathbf{e} \rangle$ for a fixed $\mathbf{e}$, we have $f(p) = \langle \mathbf{c}, \mathbf{c} \rangle - p^2 \langle \mathbf{e}, \mathbf{e} \rangle$. Therefore, $(df/dp)_{p=0} = 0$.

In dealing with cavity resonators, let $\mathbf{a}$ be an arbitrary electromagnetic field defined at the resonant frequency associated with a correct resonant field $\mathbf{c}$. According to the reciprocity theorem, therefore, we have $\langle \mathbf{a}, \mathbf{c} \rangle = 0$, which also reflects the degenerate character of the reaction. Using this fact and enforcing the condition $\langle \mathbf{a}, \mathbf{a} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle$, in an excellent chapter on variational techniques [10, Ch. 7], Harrington develops a systematic method for variational formulation of resonant frequency by setting $\langle \mathbf{a}, \mathbf{a} \rangle = 0$. According to (20) and Harrington's reasoning, $\langle \mathbf{a}, \mathbf{a} \rangle$ is stationary about the correct resonant field $\mathbf{c}$ if $\langle \mathbf{a}, \mathbf{a} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle = 0$. Therefore, by setting $\langle \mathbf{a}, \mathbf{a} \rangle = 0$, Harrington defines $\omega$ as a function of the field distribution setting up $\mathbf{a}$ and by using the stationary character of $\langle \mathbf{a}, \mathbf{a} \rangle$ about $\mathbf{c}$, he proves $\omega$ is stationary about the correct resonant field.

In our improved systematic method, we claim that $\langle \mathbf{a}, \mathbf{a} \rangle$ and its generalized form $\langle \mathbf{a}, \mathbf{a} \rangle_\alpha$ is stationary about $\mathbf{c}$ according to the following lemma.

*Fundamental Lemma:* Let $\mathbf{e}$ be an arbitrary, but fixed electromagnetic field defined at some resonant frequency of a cavity resonator and $\mathbf{c}$ be the corresponding exact resonant field. Moreover, assume that the approximate field $\mathbf{a}$ is defined as $\mathbf{a} = \mathbf{c} + p\mathbf{e}$. The function $f_\alpha(p) = \langle \mathbf{a}, \mathbf{a} \rangle_\alpha = \langle \mathbf{c} + p\mathbf{e}, \mathbf{c} + p\mathbf{e} \rangle_\alpha$ is then stationary about $p = 0$.

*Proof:* Since reaction is degenerate and bilinear, $p\mathbf{e}$ is orthogonal to $\mathbf{c}$. Therefore, according to Pythagoras' theorem and, again, bilinear character of the reaction, we have

$$\langle \mathbf{a}, \mathbf{a} \rangle_\alpha = \langle \mathbf{c} + p\mathbf{e}, \mathbf{c} + p\mathbf{e} \rangle_\alpha = \langle \mathbf{c}, \mathbf{c} \rangle_\alpha + p^2 \langle \mathbf{e}, \mathbf{e} \rangle_\alpha.$$

The above relation indicates that $\langle \mathbf{a}, \mathbf{a} \rangle_\alpha$ is stationary about $\mathbf{c}$.□

Let us see how based on the above lemma one may develop variational formulas for a cavity resonator. To this end, assume that $\mathbf{a} = \mathbf{c} + p\mathbf{e}$ for a fixed error field $\mathbf{e}$ defined at the resonant frequency and define $f_\alpha(p)$ as follows:

$$\begin{aligned} f_\alpha(p) &\triangleq \langle \mathbf{a}, \mathbf{a} \rangle_\alpha \\ &= \langle \mathbf{c}, \mathbf{c} \rangle_\alpha + \langle \mathbf{c}, p\mathbf{e} \rangle_\alpha + \langle p\mathbf{e}, \mathbf{c} \rangle_\alpha + p^2 \langle \mathbf{e}, \mathbf{e} \rangle_\alpha \\ &= p^2 \langle \mathbf{e}, \mathbf{e} \rangle_\alpha. \end{aligned} \tag{21}$$

From the above equation, we have $f_\alpha(0) = 0$ and for nonzero values of $p, f_\alpha(p) \neq 0$. On the other hand, one may write $f_\alpha(p) = F_\alpha(\omega_r, p)$, where $\omega_r$ is the resonant frequency of the cavity associated with $\mathbf{c}$. Therefore, according to (21), one may write

$$\left( \frac{df_\alpha}{dp} \right)_{p=0} = \left( \frac{\partial F_\alpha}{\partial p} \right)_{p=0} = 0. \tag{22}$$

Equation (22) is the key result for developing variational formulas for the resonant frequency. More precisely, let us define $\omega_\alpha$ as a function of $p$ by an implicit relation

$$F_\alpha [\omega_\alpha(p), p] = k \tag{23}$$

where $k$ is an arbitrary constant. Since we should have $\omega_\alpha(0) = \omega_r$ and $F_\alpha(\omega_r, 0) = f_\alpha(0) = 0$, the only permissible value of $k$ is zero. Otherwise, $\omega_\alpha(0) \neq \omega_r$ and the second equality in (22) does not hold. Moreover, since for nonzero values of $p, f_\alpha(p) = F_\alpha(\omega_r, p) \neq 0$, setting $k = 0$ in (23) guarantees that the frequency obtained for any nonzero value of $p$ is different than the resonant frequency. Now

$$\left( \frac{dF_\alpha}{dp} \right)_{p=0} = 0 = \left( \frac{\partial F_\alpha}{\partial \omega_\alpha} \frac{d\omega_\alpha}{dp} \right)_{\omega_\alpha = \omega_\alpha(0), p=0} + \left( \frac{\partial F_\alpha}{\partial p} \right)_{\omega_\alpha = \omega_\alpha(0), p=0}. \tag{24}$$

Since $\omega_\alpha(0) = \omega_r$, according to (22), $(\partial F_\alpha / \partial p)_{\omega_\alpha = \omega_\alpha(0), p=0} = 0$ and noting that, in general, $\partial F_\alpha / \partial \omega_\alpha \neq 0$, we have

$$\left( \frac{d\omega_\alpha}{dp} \right)_{p=0} = 0 \tag{25}$$

which states that $\omega_\alpha(p)$ defined by (23) with $k = 0$ is stationary about $p = 0$.

From the above considerations, one may end up with the following lemma.

*Lemma 1:* Let $\mathbf{a}$ be an arbitrary electromagnetic field defined at the resonant frequency $\omega_r$ inside a cavity resonator associated with assumed $\mathbf{E}$-field, $\mathbf{H}$-field, or mixed field ($\mathbf{E}$ and $\mathbf{H}$) distribution within the cavity. By setting $\langle \mathbf{a}, \mathbf{a} \rangle_\alpha = 0$ and changing $\omega_r$ to $\omega_\alpha$, one may then define $\omega_\alpha$ as a function of the associated vector field(s), which is stationary about the correct resonant field $\mathbf{c}$.

The key idea is that $\omega_\alpha$ is a function of the assumed field distributions setting up $\mathbf{a}$ at the resonant frequency $\omega_r$, and changing the role of $\omega_r$ to $\omega_\alpha$ does not imply that $\mathbf{a}$ is defined at some frequency other than $\omega_r$. Otherwise, the fundamental lemma (in our approach) will be invalid and the condition $\langle \mathbf{a}, \mathbf{a} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle$ (in Harrington's approach) is meaningless. In practice, one may skip the step of replacing $\omega_r$ with $\omega_\alpha$ in $\langle \mathbf{a}, \mathbf{a} \rangle_\alpha$ and set $\langle \mathbf{a}, \mathbf{a} \rangle_\alpha = 0$ as if $\mathbf{a}$ is defined at the frequency $\omega_\alpha$. Note that $\omega_\alpha \neq \omega_r$ unless $\mathbf{a} = \mathbf{c}$.

Except generalized reaction, Lemma 1 is in accordance with Harrington's formulation. However, we have proven it based on the fundamental lemma. Developing generalized reaction and fundamental lemma are the distinguishing features of this paper from Harrington's earlier work. The importance of the fundamental lemma will be clear later when we use it to generalize conventional mode-matching method discussed in Part II of this paper [12].

In the so-called $\mathbf{E}$- and $\mathbf{H}$-field formulation, Harrington obtains $\langle \mathbf{a}, \mathbf{a} \rangle$ in terms of the assumed $\mathbf{E}$- and $\mathbf{H}$-field distributions, respectively. Whereas in the mixed-field formulation $\langle \mathbf{a}, \mathbf{a} \rangle$ is obtained in terms of both $\mathbf{E}$- and $\mathbf{H}$-field distributions.

In either case, one may show that if $\mathbf{a} = \mathbf{c} + p\mathbf{e}$ for a fixed $\mathbf{e}$ defined at the resonant frequency $\omega_r$, we have

$$f(p) \triangleq \langle \mathbf{c} + p\mathbf{e}, \mathbf{c} + p\mathbf{e} \rangle \propto (j\omega_r)^n D(p) + N(p) \tag{26}$$

where $n = 2$ for $\mathbf{E}$- or $\mathbf{H}$-field formulation and $n = 1$ for mixed-field formulation. $D(p)$ and $N(p)$ are energy-type integrals and will be defined later. In either case, we have

$$f'(0) = (j\omega_r)^n D'(0) + N'(0) = 0 \tag{27}$$

or

$$\omega_r^n = -(-j)^n \frac{N'(0)}{D'(0)}. \tag{28}$$

On the other hand, $f(0) = 0 = (j\omega_r)^n D(0) + N(0)$. Therefore,

$$\omega_r^n = -(-j)^n \frac{N(0)}{D(0)}. \tag{29}$$

By using (26), if one defines $j^n \omega^n(p) D(p) + N(p) = 0$, we have $\omega(0) = \omega_r$ and

$$\omega^n(p) = -(-j)^n \frac{N(p)}{D(p)}. \tag{30}$$

Now one may easily show that $(d\omega^n/dp)_{p=0} = 0$. This fact can be argued by noting that, from (28) and (29), we have $N'(0)/D'(0) = N(0)/D(0)$ and, therefore, $N'(0)D(0) - D'(0)N(0) = 0$, which means that $(d\omega^n/dp)_{p=0} = 0$. This result is expected and we have just used a more straightforward way to prove the stationary character of $\omega^n(p)$ about $p = 0$, which was proven earlier by considering a more general case. Recall that, according to the general case, by taking the derivative of the implicit relation $j^n \omega^n(p) D(p) + N(p) = 0$ at $p = 0$, we will have $nj^n \omega^{n-1}(0)\omega'(0)D(0) + j^n \omega^n(0) D'(0) + N'(0) = 0$ or $nj^n \omega_r^{n-1} \omega'(0) D(0) + (j\omega_r)^n D'(0) + N'(0) = 0$. Using (27) and noting that, in general, $D(0) \neq 0$, we have $\omega'(0) = 0$.

Note that $D(p)$ and $N(p)$ are defined based on the conventional reaction. Explicit relations for $D(p)$ and $N(p)$ and their generalized forms $D_\alpha(p)$ and $N_\alpha(p)$ will be given later when we derive unrestricted variational formulas of the cavity resonator shown in Fig. 1.

The Rayleigh–Ritz approach is a powerful method, which usually comes with the variational formulation. Harrington, by using the stationary character of $\omega^n(p)$, in a typical example [10, pp. 339–340] tries to apply the Rayleigh–Ritz method in a cavity resonator by considering only two basis functions for the field expansion. Due to the quadratic nature of the equations, the unknown expansion coefficients and resonant frequencies are obtained by solving a system of nonlinear equations. Of course, this approach cannot be used if one expands the fields in terms of more than two basis functions. However, (27) is a derivative of a quadratic equation, which leads to a system of linear equations and is more suitable for the Rayleigh–Ritz approach. More precisely, let $\mathbf{c} = \sum_{n=1}^{N} A_n \mathbf{v}_n$, where $A_n$'s are the unknown expansion coefficients. Assume that one of these coefficients, e.g., $A_i$ is changed to $A_i + p$. Therefore, the correct field $\mathbf{c}$ reduces to the approximate field $\mathbf{a}$ such that $\mathbf{a} = \mathbf{c} + p\mathbf{v}_i$. Treating $\mathbf{v}_i$ as an error field, one may see that (26) and (27) imply

$$\begin{aligned} 0 &= \left[ \frac{d\langle \mathbf{c} + p\mathbf{v}_i, \mathbf{c} + p\mathbf{v}_i \rangle}{dp} \right]_{p=0} \\ &= \lim_{p \to 0} \frac{\langle \mathbf{c} + p\mathbf{v}_i, \mathbf{c} + p\mathbf{v}_i \rangle - \langle \mathbf{c}, \mathbf{c} \rangle}{p} \\ &= \frac{\partial \langle \mathbf{c}, \mathbf{c} \rangle}{\partial A_i} \triangleq \left[ \frac{\langle \mathbf{a}, \mathbf{a} \rangle}{\partial A_i} \right]_{p=0} \end{aligned} \tag{31}$$

where we have used $\mathbf{c} + p\mathbf{v}_i = \sum_{n \neq i}^{N} A_n \mathbf{v}_n + (A_i + p)\mathbf{v}_i$. Moreover, as indicated in (31), $\langle \mathbf{c}, \mathbf{c} \rangle$ is obtained from $\langle \mathbf{a}, \mathbf{a} \rangle$ by replacing $\mathbf{E}^a$ and/or $\mathbf{H}^a$ with $\mathbf{E}^c$ and/or $\mathbf{H}^c$, respectively. The condition $[(d/dp)\langle \mathbf{c} + p\mathbf{v}_i, \mathbf{c} + p\mathbf{v}_i \rangle]_{p=0} = 0, (i = 1, 2, \ldots, N)$ or its equivalent $(\partial/\partial A_i)\langle \mathbf{c}, \mathbf{c} \rangle = 0, (i = 1, 2, \ldots, N)$ implies $\langle \mathbf{c}, \mathbf{v}_i \rangle + \langle \mathbf{v}_i, \mathbf{c} \rangle = 0, (i = 1, 2, \ldots, N)$. Using the symmetric character of the reaction, one may end up with $2\langle \mathbf{c}, \mathbf{v}_i \rangle = 0, (i = 1, 2, \ldots, N)$ or

$$\left\langle \sum_{n=1}^{N} A_n \mathbf{v}_n, \mathbf{v}_i \right\rangle = \sum_{n=1}^{N} A_n \langle \mathbf{v}_n, \mathbf{v}_i \rangle = 0, \qquad i = 1, 2, \ldots, N \tag{32}$$

which leads to a system of $N$ homogeneous linear equations for the unknown expansion coefficients $A_n$'s. From the above considerations, one may end up with two different, but equivalent approaches in setting up a system of $N$ homogeneous linear equations in $A_n$'s. In the first approach the linear equations are of the form $(\partial/\partial A_i)\langle \mathbf{c}, \mathbf{c} \rangle = 0, (i = 1, 2, \ldots, N)$, whereas in the second one, we have $\sum_{n=1}^{N} A_n \langle \mathbf{v}_n, \mathbf{v}_i \rangle = 0, (i = 1, 2, \ldots, N)$. However, the former enjoys the advantage that it eliminates the need of finding the sources of $\mathbf{v}_n$'s. The reader may verify the validity of the above assertions by considering the aforementioned example treated by Harrington and applying (31) or (32) to obtain the unknown expansion coefficients and the resonant frequencies by a matrix equation. The resonant frequencies can be obtained by seeking the zeros of the determinant of the coefficient matrix.

The above are the fundamental principles of variational formulation in cavity resonators. The restrictions set forth on the class of trial fields depend on the generality of the reaction. The more general the definition of the reaction, the less restrictions on the trial fields. Since the reaction defined in the literature requires at least the tangential components of the electric or magnetic field be continuous on any interface, the stationary formulas for the resonant frequency requires at least $\mathbf{E}^a$ or $\mu^{-1}\nabla \times \mathbf{E}^a$ in $\mathbf{E}$-field formulation, $\mathbf{H}^a$ or $\epsilon^{-1}\nabla \times \mathbf{H}^a$ in $\mathbf{H}$-field formulation, and $\mathbf{E}^a$ or $\mathbf{H}^a$ in the mixed-field formulation satisfy the boundary conditions at any interface within the cavity.

Since we have generalized the reaction that relaxes the above restrictions, one may expect to obtain unrestricted stationary formulas for the resonant frequency. Suppose that we are interested in obtaining the $\mathbf{E}$-field variational formula for the resonant frequency in the cavity resonator shown in Fig. 1 without any restriction on the trial fields. To this end, assume that $\mathbf{E}^a$ is an arbitrary vector field defined inside the cavity. The electromagnetic field $\mathbf{a}$ set up by $\mathbf{E}^a$ at the resonant frequency $\omega_r$ can be defined as $\mathbf{H}^a_{1,2} = j\omega_r^{-1}\mu_{1,2}^{-1}\nabla\times\mathbf{E}^a_{1,2}$ and $\mathbf{J}^a_{1,2} = -j\omega_r\epsilon_{1,2}\mathbf{E}^a_{1,2} + j\omega_r^{-1}\nabla\times\mu_{1,2}^{-1}\nabla\times\mathbf{E}^a_{1,2}$. Including the necessary surface currents to support the discontinuities across the boundaries and using the generalized reaction yield

$$\langle\mathbf{a},\mathbf{a}\rangle_\alpha = \frac{j}{\omega_r}\Big[-\omega_r^2 D^e_\alpha(\mathbf{E}^a) + N^e_\alpha(\mathbf{E}^a)\Big] \tag{33}$$

where

$$\begin{aligned} N^e_\alpha(\mathbf{E}^a) = &\int_{V_1+V_2}\mu_{1,2}^{-1}\Big(\nabla\times\mathbf{E}^a_{1,2}\Big)\cdot\Big(\nabla\times\mathbf{E}^a_{1,2}\Big)dv\\ &+2\int_{\Sigma_e}\mu_1^{-1}\nabla\times\mathbf{E}^a_1\times\mathbf{E}^a_1\cdot\hat{\mathbf{n}}_\Sigma\,ds\\ &+2\int_{\Sigma_e}\mu_2^{-1}\nabla\times\mathbf{E}^a_2\times\mathbf{E}^a_2\cdot(-\hat{\mathbf{n}}_\Sigma)\,ds\\ &+2\int_{S_e}\mu_2^{-1}\nabla\times\mathbf{E}^a_2\times\mathbf{E}^a_2\cdot\hat{\mathbf{n}}_S\,ds\\ &+\int_{\Sigma_i}\Big(\mu_1^{-1}\nabla\times\mathbf{E}^a_1+\mu_2^{-1}\nabla\times\mathbf{E}^a_2\Big)\\ &\times\Big(\mathbf{E}^a_1-\mathbf{E}^a_2\Big)\cdot\hat{\mathbf{n}}_\Sigma\,ds-\int_{\Sigma_i}\big[2\alpha(\sigma)-1\big]\\ &\times\Big(\mu_1^{-1}\nabla\times\mathbf{E}^a_1-\mu_2^{-1}\nabla\times\mathbf{E}^a_2\Big)\\ &\times\Big(\mathbf{E}^a_1-\mathbf{E}^a_2\Big)\cdot\hat{\mathbf{n}}_\Sigma\,ds \end{aligned}\tag{34}$$

$$D^e_\alpha(\mathbf{E}^a) = \int_{V_1+V_2}\epsilon_{1,2}\mathbf{E}^a_{1,2}\cdot\mathbf{E}^a_{1,2}\,dv \tag{35}$$

and we have used the identity $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A}-\mathbf{A}\cdot\nabla\times\mathbf{B}$. Note that $D^e_\alpha(\mathbf{E}^a)$ is independent of $\alpha$.

By setting $\langle\mathbf{a},\mathbf{a}\rangle_\alpha = 0$ and changing $\omega_r$ to $\omega_\alpha, \omega_\alpha$ is defined as a function of $\mathbf{E}^a$ by an implicit relation $-\omega_\alpha^2 D^e_\alpha(\mathbf{E}^a) + N^e_\alpha(\mathbf{E}^a) = 0$ or explicitly as

$$\omega_\alpha^2 = \frac{N^e_\alpha(\mathbf{E}^a)}{D^e_\alpha(\mathbf{E}^a)}. \tag{36}$$

To prove (36) is stationary about the correct resonant field, let $\mathbf{E}^a = \mathbf{E}^c + p\mathbf{E}^e$, where $\mathbf{E}^c$ is the electric-field distribution of the correct resonant field $\mathbf{c}$ and $\mathbf{E}^e$ is an arbitrary, but fixed vector field defined within the cavity. Substituting for $\mathbf{E}^a$ into (36), one may define $\omega_\alpha^2$ as a function of $p$ as follows:

$$\omega_\alpha^2(p) = \frac{N_\alpha(p)}{D_\alpha(p)} \tag{37}$$

where $N_\alpha(p) \triangleq N^e_\alpha(\mathbf{E}^c + p\mathbf{E}^e)$ and $D_\alpha(p) \triangleq D^e_\alpha(\mathbf{E}^c + p\mathbf{E}^e)$. According to the earlier discussions, vanishing the derivative of (37) at $p = 0$ is a sufficient condition for $\omega_\alpha^2$, defined by (36), to be stationary about the correct resonant field distribution $\mathbf{E}^c$. To this end, we define the electromagnetic field $\mathbf{e}$ set up by $\mathbf{E}^e$ at $\omega_r$ as $\mathbf{H}^e_{1,2} = j\omega_r^{-1}\mu_{1,2}^{-1}\nabla\times\mathbf{E}^e_{1,2}$ and $\mathbf{J}^e_{1,2} = -j\omega_r\epsilon_{1,2}\mathbf{E}^e_{1,2} + j\omega_r^{-1}\nabla\times\mu_{1,2}^{-1}\nabla\times\mathbf{E}^e_{1,2}$. By including the necessary surface currents to support the discontinuities of $\mathbf{e}$ across the boundaries and using the generalized reaction, we obtain

$$f_\alpha(p) = \langle\mathbf{c}+p\mathbf{e},\mathbf{c}+p\mathbf{e}\rangle_\alpha = \frac{j}{\omega_r}\big[-\omega_r^2 D_\alpha(p) + N_\alpha(p)\big]. \tag{38}$$

Now $f_\alpha(0) = 0 = -\omega_r^2 D_\alpha(0) + N_\alpha(0)$ and, therefore, $\omega_r^2 = N_\alpha(0)/D_\alpha(0) = \omega_\alpha(0)$. Moreover, according to the fundamental lemma, $f'_\alpha(0) = 0 = -\omega_r^2 D'_\alpha(0) + N'_\alpha(0)$, which indicates that $\omega_r^2 = N'_\alpha(0)/D'_\alpha(0)$. Therefore, $N_\alpha(0)/D_\alpha(0) = N'_\alpha(0)/D'_\alpha(0)$ and $\omega^2(p)$ defined by (37) is stationary about $p = 0$. It should be noted that the stationary character of (36) can be proven directly by showing that

$$-\omega_r^2\left[\frac{d}{dp}D^e_\alpha(\mathbf{E}^c+p\mathbf{E}^e)\right]_{p=0} + \left[\frac{d}{dp}N^e_\alpha(\mathbf{E}^c+p\mathbf{E}^e)\right]_{p=0} = 0. \tag{39}$$

To prove (39) directly, one should use the identity $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A}-\mathbf{A}\cdot\nabla\times\mathbf{B}$ and note that $-\omega_r^2 D^e_\alpha(\mathbf{E}^c) + N^e_\alpha(\mathbf{E}^c) = 0$.

An interesting feature of the unrestricted variational formulation of cavity resonators is its dependence on $\alpha(\sigma)$ and, in other words, its nonuniqueness character. As can be read from (34), the term $I^e_\alpha(\mathbf{E}^a) \triangleq \int_{\Sigma_i}[2\alpha(\sigma)-1](\mu_1^{-1}\nabla\times\mathbf{E}^a_1 - \mu_2^{-1}\nabla\times\mathbf{E}^a_2)\times(\mathbf{E}^a_1-\mathbf{E}^a_2)\cdot\hat{\mathbf{n}}_\Sigma\,ds$ depends on $\alpha(\sigma)$ and the integrand is proportional to the products of the differences of fields on both sides of the interface. In fact, by defining $I_\alpha(p) \triangleq I^e_\alpha(\mathbf{E}^c + p\mathbf{E}^e)$, where $\mathbf{E}^e$ is an arbitrary, but fixed vector field, and noting that the correct resonant field satisfies the boundary conditions, we have $I_\alpha(p) = p^2\int_{\Sigma_i}[2\alpha(\sigma)-1](\mu_1^{-1}\nabla\times\mathbf{E}^e_1 - \mu_2^{-1}\nabla\times\mathbf{E}^e_2)\times(\mathbf{E}^e_1-\mathbf{E}^e_2)\cdot\hat{\mathbf{n}}_\Sigma\,ds$. Therefore, $I_\alpha(0) = I'_\alpha(0) = 0$ for any $\alpha(\sigma)$, and since $I^e_\alpha(\mathbf{E}^a)$ is an additive term in $N^e_\alpha(\mathbf{E}^a)$, (39) also holds for any $\alpha(\sigma)$. If $\alpha(\sigma) \equiv 1/2$ or, if the tangential components of either $\mathbf{E}^a$ or $\mu^{-1}\nabla\times\mathbf{E}^a$ is continuous across $\Sigma_i,\ I^e_\alpha(\mathbf{E}^a)$ vanishes. It is also interesting to note that, in the $\mathbf{E}$-field formulation, as expressed by (36), no surface integral is present over the surface of a perfect magnetic conductor. As expected, for restricted field distributions, this generalized variational formulation reduces to those special ones given by Harrington. Strictly speaking, let $\mathcal{A}(\cdot)$ be an operator when acting on a surface gives its area. It can be seen that under special cases where $\mathcal{A}(\Sigma_e) = \mathcal{A}(\Sigma_m) = \mathcal{A}(S_m) = 0$ and $\mathbf{E}^a$ or $\mu^{-1}\nabla\times\mathbf{E}^a$ meets the continuity conditions across $\Sigma_i$, the variational formulas reduce to those given by Harrington. Under these special conditions, $D(p)$ and $N(p)$, used in our earlier discussion, will be the reduced forms of $D_\alpha(p)$ and $N_\alpha(p)$, respectively.

In a similar fashion, the $\mathbf{H}$-field variational formulation of the cavity resonator shown in Fig. 1 can be obtained without any restriction on the trial fields. Let $\mathbf{H}^a$ be an arbitrary vector field, which sets up the electromagnetic field $\mathbf{a}$ at the resonant frequency $\omega_r$ by $\mathbf{E}^a_{1,2} = -j\omega_r^{-1}\epsilon_{1,2}^{-1}\nabla\times\mathbf{H}^a_{1,2}$ and $\mathbf{M}^a_{1,2} = -j\omega_r\mu_{1,2}\mathbf{H}^a_{1,2} + j\omega_r^{-1}\nabla\times\epsilon_{1,2}^{-1}\nabla\times\mathbf{H}^a_{1,2}$. By including the necessary surface currents to support the discontinuities across the boundaries and obtaining $\langle\mathbf{a},\mathbf{a}\rangle_\alpha$, setting $\langle\mathbf{a},\mathbf{a}\rangle_\alpha = 0$, and changing $\omega_r$ to $\omega_\alpha$, the unrestricted variational formula for the resonant frequency becomes

$$\omega_\alpha^2 = \frac{N_\alpha^h(\mathbf{H}^a)}{D_\alpha^h(\mathbf{H}^a)} \tag{40}$$

where

$$\begin{aligned} N_\alpha^h(\mathbf{H}^a) = &\int_{V_1+V_2} \epsilon_{1,2}^{-1}\Big(\nabla\times\mathbf{H}_{1,2}^a\Big)\cdot\Big(\nabla\times\mathbf{H}_{1,2}^a\Big)dv \\ &+2\int_{\Sigma_m}\epsilon_1^{-1}\nabla\times\mathbf{H}_1^a\times\mathbf{H}_1^a\cdot\hat{\mathbf{n}}_\Sigma\,ds \\ &+2\int_{\Sigma_m}\epsilon_2^{-1}\nabla\times\mathbf{H}_2^a\times\mathbf{H}_2^a\cdot(-\hat{\mathbf{n}}_\Sigma)\,ds \\ &+2\int_{S_m}\epsilon_2^{-1}\nabla\times\mathbf{H}_2^a\times\mathbf{H}_2^a\cdot\hat{\mathbf{n}}_S\,ds \\ &+\int_{\Sigma_i}\Big(\epsilon_1^{-1}\nabla\times\mathbf{H}_1^a+\epsilon_2^{-1}\nabla\times\mathbf{H}_2^a\Big) \\ &\times\Big(\mathbf{H}_1^a-\mathbf{H}_2^a\Big)\cdot\hat{\mathbf{n}}_\Sigma\,ds+\int_{\Sigma_i}\big[2\alpha(\sigma)-1\big] \\ &\times\Big(\epsilon_1^{-1}\nabla\times\mathbf{H}_1^a-\epsilon_2^{-1}\nabla\times\mathbf{H}_2^a\Big) \\ &\times\Big(\mathbf{H}_1^a-\mathbf{H}_2^a\Big)\cdot\hat{\mathbf{n}}_\Sigma\,ds \end{aligned} \tag{41}$$

$$D_\alpha^h(\mathbf{H}^a) = \int_{V_1+V_2}\mu_{1,2}\mathbf{H}_{1,2}^a\cdot\mathbf{H}_{1,2}^a\,dv \tag{42}$$

In the $\mathbf{H}$-field formulation, as expressed by (40)–(42), no surface integral is present over the surface of a perfect electric conductor, and like the $\mathbf{E}$-field formulation, the $\mathbf{H}$-field formulation is not unique and depends on $\alpha(\sigma)$. It can also be seen that, in special restricted cases, the generalized $\mathbf{H}$-field formulas reduce to those given by Harrington. It is interesting to note that duality applies to $\mathbf{E}$-field and $\mathbf{H}$-field formulations of the cavity shown in Fig. 1. More precisely, the $\mathbf{H}$-field formulation can be obtained from the $\mathbf{E}$-field formulation by the following transformations:

$$\begin{aligned} \mathbf{E}_{1,2}^a &\to \mathbf{H}_{1,2}^a \\ \mu_{1,2} &\to \epsilon_{1,2} \\ \Sigma_e &\to \Sigma_m \\ S_e &\to S_m \\ \alpha(\sigma) &\to 1-\alpha(\sigma). \end{aligned}$$

In a dual fashion, the $\mathbf{E}$-field formulation can be obtained from the $\mathbf{H}$-field formulation by the following transformations:

$$\begin{aligned} \mathbf{H}_{1,2}^a &\to \mathbf{E}_{1,2}^a \\ \epsilon_{1,2} &\to \mu_{1,2} \\ \Sigma_m &\to \Sigma_e \\ S_m &\to S_e \\ \alpha(\sigma) &\to 1-\alpha(\sigma). \end{aligned}$$

It should be emphasized that duality does not apply to $\mathbf{E}$-field and $\mathbf{H}$-field formulations if the cavity walls or its interior boundaries are made of only one type of perfect conductor.

To obtain the mixed-field variational formula, we assume arbitrary vector fields $\mathbf{E}^a$ and $\mathbf{H}^a$ within the cavity shown in Fig. 1. These vector fields define the electromagnetic field $\mathbf{a}$ at the resonant frequency $\omega_r$ if one considers $\mathbf{M}_{1,2}^a = -\nabla\times\mathbf{E}_{1,2}^a - j\omega_r\mu_{1,2}\mathbf{H}_{1,2}^a, \mathbf{J}_{1,2}^a = \nabla\times\mathbf{H}_{1,2}^a - j\omega_r\epsilon_{1,2}\mathbf{E}_{1,2}^a$, and the necessary surface currents to support the discontinuities across the boundaries. Therefore, with this procedure, the generalized reaction can be obtained in terms of arbitrary vector fields $\mathbf{E}^a$ and $\mathbf{H}^a$ as follows:

$$\langle\mathbf{a},\mathbf{a}\rangle_\alpha = j\omega_r D_\alpha^{eh}\big(\mathbf{E}^a,\mathbf{H}^a\big) + N_\alpha^{eh}\big(\mathbf{E}^a,\mathbf{H}^a\big) \tag{43}$$

where

$$\begin{aligned} &N_\alpha^{eh}(\mathbf{E}^a,\mathbf{H}^a) \\ &\quad=\int_{V_1+V_2}\Big(\mathbf{E}_{1,2}^a\cdot\nabla\times\mathbf{H}_{1,2}^a+\mathbf{H}_{1,2}^a\cdot\nabla\times\mathbf{E}_{1,2}^a\Big)dv \\ &\quad\quad+\int_{\Sigma_m}\Big(\mathbf{E}_1^a\times\mathbf{H}_1^a-\mathbf{E}_2^a\times\mathbf{H}_2^a\Big)\cdot\hat{\mathbf{n}}_\Sigma\,ds \\ &\quad\quad-\int_{\Sigma_e}\Big(\mathbf{E}_1^a\times\mathbf{H}_1^a-\mathbf{E}_2^a\times\mathbf{H}_2^a\Big)\cdot\hat{\mathbf{n}}_\Sigma\,ds \\ &\quad\quad+\int_{S_m}\mathbf{E}_2^a\times\mathbf{H}_2^a\cdot\hat{\mathbf{n}}_S\,ds-\int_{S_e}\mathbf{E}_2^a\times\mathbf{H}_2^a\cdot\hat{\mathbf{n}}_S\,ds \\ &\quad\quad+\int_{\Sigma_i}\Big(\mathbf{E}_2^a\times\mathbf{H}_1^a-\mathbf{E}_1^a\times\mathbf{H}_2^a\Big)\cdot\hat{\mathbf{n}}_\Sigma\,ds \\ &\quad\quad+\int_{\Sigma_i}\big[2\alpha(\sigma)-1\big]\Big(\mathbf{E}_1^a-\mathbf{E}_2^a\Big)\times\Big(\mathbf{H}_1^a-\mathbf{H}_2^a\Big)\cdot\hat{\mathbf{n}}_\Sigma\,ds \end{aligned} \tag{44}$$

$$\begin{aligned} &D_\alpha^{eh}(\mathbf{E}^a,\mathbf{H}^a) \\ &\quad=\int_{V_1+V_2}\Big(\mu_{1,2}\mathbf{H}_{1,2}^a\cdot\mathbf{H}_{1,2}^a-\epsilon_{1,2}\mathbf{E}_{1,2}^a\cdot\mathbf{E}_{1,2}^a\Big)dv. \end{aligned} \tag{45}$$

Following the same reasoning used in the $\mathbf{E}$- and $\mathbf{H}$-field formulations, it can be shown that

$$\omega_\alpha = j\frac{N_\alpha^{eh}\big(\mathbf{E}^a,\mathbf{H}^a\big)}{D_\alpha^{eh}\big(\mathbf{E}^a,\mathbf{H}^a\big)} \tag{46}$$

is stationary about the correct resonant field. In this case, by fixing $\mathbf{E}^e$ and $\mathbf{H}^e$ and defining $N_\alpha(p) \triangleq N_\alpha^{eh}(\mathbf{E}^c+p\mathbf{E}^e,\mathbf{H}^c+p\mathbf{H}^e)$ and $D_\alpha(p) \triangleq D_\alpha^{eh}(\mathbf{E}^c+p\mathbf{E}^e,\mathbf{H}^c+p\mathbf{H}^e)$, it suffices to show that

$$\omega_\alpha(p) = j\frac{N_\alpha(p)}{D_\alpha(p)} \tag{47}$$

is stationary about $p=0$.

In this section, based on the degenerate character of generalized reaction, we have developed a systematic method for obtaining unrestricted variational formulas for cavity resonators and derived various variational formulas for an arbitrary, but not simple resonant structure shown in Fig. 1.

## IV. Numerical Results

In this section, we apply the formulation developed in the preceding section to some physical structures. In all formulas, we set $\alpha \equiv 1/2$. In the first example, we obtain the approximate resonant frequencies of a rectangular cavity with a cylindrical dielectric rod in it. The length of the rod covers the full height of the cavity with the cross section shown in Fig. 2. We have

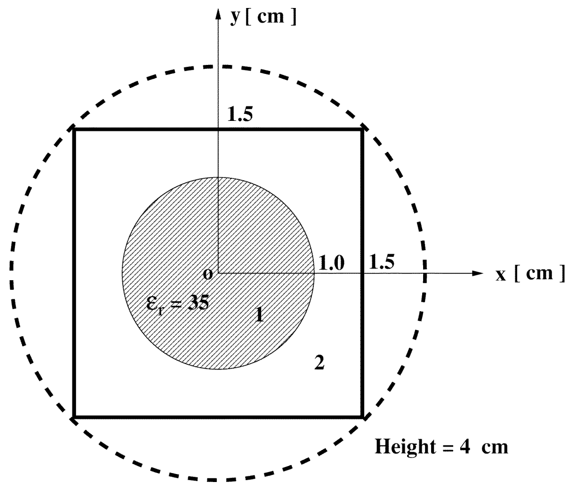


Fig. 2. Cross sections of rectangular and cylindrical cavity resonators with a cylindrical dielectric rod inside them.

TABLE I
COMPARISON OF THE RESONANT FREQUENCIES (IN GIGAHERTZ) OF A RECTANGULAR CAVITY OBTAINED BY DIFFERENT APPROXIMATE METHODS

| Mode | HFSS | Cylinder | **E**-field | **H**-field | Mixed field | $\text{Error}_{\text{cylinder}}$ | $\text{Error}_{\textbf{H}\text{-field}}$ |
|---|---|---|---|---|---|---|---|
| $TM_0$ | 1.303 | 1.1210 | 1.2433 | 1.2778 | 1.2668 | -14.0% | -2.0% |
| $TE_0$ | 2.330 | 2.2330 | 2.2880 | 2.2908 | 2.2900 | -4.2% | -1.7% |
| $HE_{11}$ | 2.0145 | 1.939 | 1.9504 | 2.0303 | 1.9896 | -3.7% | 0.8% |

used the resonant fields of a cylindrical cavity surrounding the rectangular one as the trial fields. The cross section of the cylindrical cavity is also shown in Fig. 2. The results for the $\mathrm{TM}_0$, $\mathrm{TE}_0$, and $\mathrm{HE}_{11}$ modes are shown in Table I.

We have also included the simulation results obtained by Ansoft HFSS as a reference. The last two columns show the relative errors of the cylindrical cavity approximation and the $\mathbf{H}$-field formulation with respect to the HFSS results, respectively.

Since the dielectric rod covers the full height of the cylindrical cavity, this cavity becomes a two-dimensional resonator for the $\mathrm{TM}_0$ mode, and this mode is the fundamental mode of the cavity. For this special structure, the cylindrical surface $r = 1$ is equivalent to the surface $\Sigma_i$ in the general cavity structure shown in Fig. 1. However, since the dielectric rod covers the full height of the cavity, the trial fields obtained by the cylindrical cavity approximation meet the continuity conditions on $\Sigma_i$. Therefore, the surface integral over $\Sigma_i$ vanishes. Moreover, in the $\mathbf{H}$-field formulation, as expressed by (40)–(42), no surface integral is present over the surface of a perfect electric conductor and, hence, over the cavity walls shown in Fig. 2. Therefore, the $\mathbf{H}$-field formulation reduces to

$$\omega_h^2 = \omega_{\mathrm{cyl}}^2 \frac{W_e}{W_h}$$

where $\omega_{\mathrm{cyl}}$ is the frequency obtained by the mode-matching method in the cylindrical cavity. $W_e$ and $W_h$ are the total electric and total magnetic energies of the trial fields in the rectangular cavity, respectively. For all modes, because of the presence of the dielectric rod of high dielectric constant, the energy stored in the electric field is highly confined within the rod and only a few percent of the electric energy is outside of it. However, this is not the case for the energy stored in the magnetic field. Therefore, $W_e > W_h$ and $\omega_h > \omega_{\mathrm{cyl}}$. For these three different cases, the $\mathbf{H}$-field formulation gives the best results and the results obtained by the mixed-field formulation are also better than those obtained by the $\mathbf{E}$-field formulation.

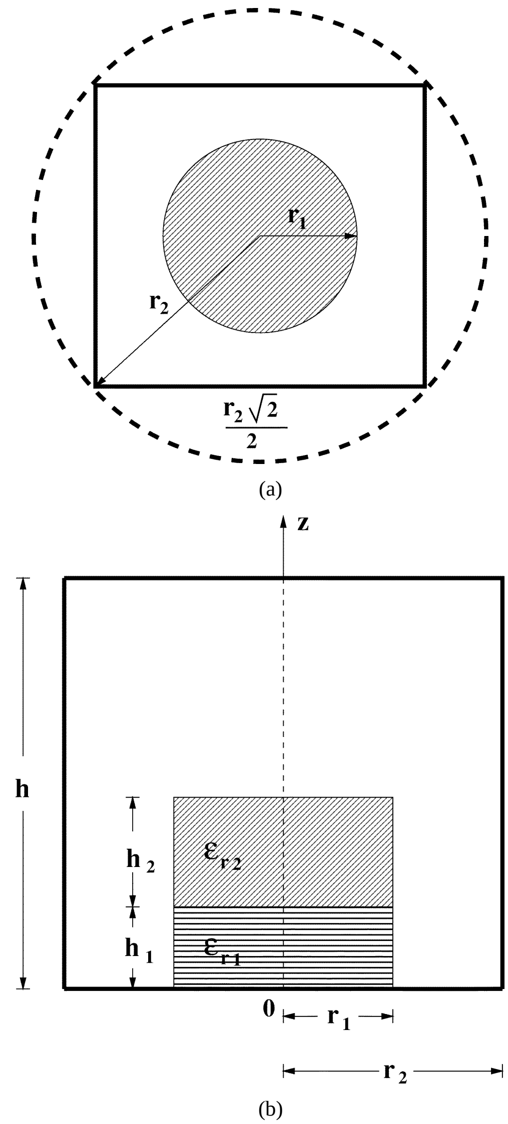


Fig. 3. DR with a support inside a: (a) rectangular cavity and (b) cylindrical cavity.

In the second example, we have considered a DR with a support inside a rectangular box. The radius of the support is the same as that of the resonator. The top view of the resonator is illustrated in Fig. 3(a). Like the previous example, we have used the resonant fields of a cylindrical cavity surrounding the rectangular box as the trial ones. The cross section of the cylindrical cavity is shown in Fig. 3(b). The trial fields have been obtained via the radial mode-matching method [4], [6]. In the radial mode-matching method, the cylindrical cavity is divided into two radial parallel-plate waveguides $0 < r < r_1$ and $r_1 < r < r_2$. We then expand the fields in terms of the modes of each section and, by matching the boundary conditions at $r = r_1$, the resonant frequencies and the unknown coefficient of each mode in the field expansion can be obtained by a matrix equation.

TABLE II
VARIOUS DIMENSIONS (IN CENTIMETERS) AND DIELECTRIC CONSTANTS OF A DR AND A SUPPORT INSIDE A RECTANGULAR CAVITY

| $r_1$ | $r_2$ | $h_1$ | $h_2$ | $h$ | $\epsilon_{r1}$ | $\epsilon_{r2}$ |
|---|---|---|---|---|---|---|
| 0.8305 | 1.795 | 0.6985 | 0.5537 | 2.3368 | 1.0 | 38.0 |

TABLE III
COMPARISON OF THE RESONANT FREQUENCIES (IN GIGAHERTZ) OF A SUPPORTED DR INSIDE A RECTANGULAR CAVITY SHOWN IN FIG. 3 & THAT ARE OBTAINED BY DIFFERENT APPROXIMATE METHODS

| Number of modes | Cylinder | **E**-field | **H**-field | Mixed-field | $\text{Error}_{\text{cylinder}}$ | $\text{Error}_{\mathbf{H}\text{-field}}$ |
|---|---|---|---|---|---|---|
| 1 | 3.4074 | 3.4593 | 3.4790 | 3.4697 | -4.6% | -2.59% |
| 2 | 3.4252 | 3.4734 | 3.4841 | 3.4791 | -4.10% | -2.45% |
| 4 | 3.4627 | 3.5072 | 3.5090 | 3.5084 | -3.05% | -1.75% |
| 8 | 3.4659 | 3.5102 | 3.5114 | 3.5111 | -2.96% | -1.69% |

It should be emphasized that the tangential components of the electric and magnetic fields obtained by a finite number of modes in the radial mode-matching method cannot be continuous on the surface $r = r_1$. Therefore, this surface is equivalent to $\Sigma_i$ in the general model of the cavity resonator illustrated in Fig. 1. Similarly, if one uses a finite number of modes in the axial mode-matching method [4], [5] to obtain the resonant fields of the cylindrical cavity, the tangential components of the fields will be discontinuous along the axis of the cavity at $z = h_1$ and $z = h_1 + h_2$. Therefore, the conventional restricted variational formulas cannot be applied in this case and only unrestricted variational formulas developed in this paper are applicable.

Various dimensions of the cavity and the dielectric constants of the materials inside it are illustrated in Table II. Unlike the previous example, the $\text{TE}_0$ mode is the lowest resonant mode of this structure. Therefore, we have considered only this mode.

The results for $\text{TE}_0$ modes are shown in Table III. To obtain the trial fields, we have used the same number of modes in radial waveguides $r < r_1$ and $r_1 < r < r_2$ inside the cylindrical cavity. The number of modes used to obtain the trial fields in each radial waveguide is also shown in Table III. The frequency obtained by Ansoft HFSS is 3.5718 GHz and we use it as a reference. The last two columns show, respectively, the relative errors of the cylindrical cavity approximation and the $\mathbf{H}$-field formulation with respect to the HFSS result.

As expected, the resonant frequencies obtained by the variational formulation are more accurate than those obtained by the cylindrical cavity approximation. Like the previous example, the $\mathbf{H}$-field formulation gives the best results and the results obtained by the mixed-field formulation are also better than those obtained by the $\mathbf{E}$-field formulation.

Other higher order modes like $\text{TM}_0$ and $\text{HE}_{11}$ can be treated in a similar fashion.

## V. CONCLUSION

In this paper, we have developed a systematic method for obtaining unrestricted variational expressions in cavity resonators. The keystone in this development is the generalized reaction and its degenerate property. Based on this formulation, the trial electric or magnetic field is not required to satisfy any boundary conditions inside a cavity. In other words, unlike existing formulas in the literature, the stationary character of the variational expressions is independent of the behavior of the electric and magnetic fields inside a cavity. Another distinguishing feature of this new formulation is its nonuniqueness character, if the tangential components of both electric and magnetic fields are discontinuous across some boundaries inside the cavity. All these formulas reduce to those conventional ones in the literature if at least the electric or magnetic field satisfies the boundary conditions inside the cavity.

The applicability of the formulas that give the approximate frequency in terms of the trial fields becomes limited if the complexity of the structure increases. In such cases, using the Rayleigh–Ritz method is more practical. Under special cases where the trial fields are not supported by any volume sources, the frequency is not present explicitly in the variational formulas. Application of the Rayleigh–Ritz method to a special form of these so-called implicit formulas with $\alpha = 1/2$ results in the generalized mode-matching method. In this generalized formulation, one may relax the orthogonality and some specific boundary conditions that should be met by the basis functions across some surfaces inside the cavity. Hence, we call it a nonorthogonal and free-boundary mode-matching method. The details of this new formulation is addressed in Part II of this paper [12].


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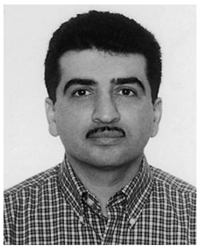

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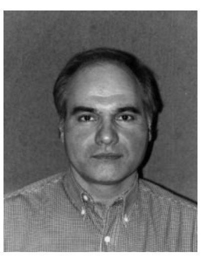

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